Int. J. Solids Structures Vol. 29, No. 23, pp. 3025–3043, 1992
Printed in Great Britain.

0020–7683/92 $5.00 + .00
Pergamon Press Ltd

A CONSTITUTIVE MODELLING OF CENTROSYMMETRIC AND NON-CENTROSYMMETRIC ANISOTROPIC FRICTION

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(*Received* 1 *February* 1991; *in revised form* 5 *August* 1991)

Abstract—A constitutive equation of anisotropic dry friction with tensors depending on a sliding direction is formulated. It describes centrosymmetric and non-centrosymmetric anisotropic friction. Symmetries, principal, neutral and extremal value directions of anisotropic friction are investigated.

1. INTRODUCTION

There has been little progress towards a general model for friction. The primary reason for this is the complex nature of friction. Descriptions applied in classical mechanics are very simple and they do not include fundamental experimental facts. It is no wonder that, for engineering practice, experimental investigations have been developed in which real problems are considered. These investigations are very useful, but it is difficult to form a general picture of the phenomenon and to introduce a generalization of observations. The second approach is a product of fundamental investigations in the physics and chemistry of surfaces. These works explain the mechanisms of the phenomenon, but they are very often not suitable for technological applications.

The best we can do is to study friction at various stages and from this reconstruct a picture of what is going on. The research needed to understand and predict the phenomenon require both mathematical and experimental modelling.

Mathematical models are defined by equations and definitions of a physical character of variables and their relations to observable facts. They describe relations which are valid for many bodies and contact conditions. This generality and a strong experimental basis are the main advantages of the mathematical modelling. A simplicity in the mathematical sense is an important criterion in the selection of a model. For simple models we can easily investigate their properties and use them to interpret the experimental results.

With the aid of mathematical models we can formulate new problems and solve them mathematically, we can plan new experiments to verify the model and if it is positive, we can apply the model to solve problems which are important for the advancement of technology. Mathematical models provide a framework for the analysis, design and optimization of mechanical systems operating in friction conditions. They give us a chance to exploit the positive effects of friction and to avoid the negative effects at the design stage.

We pay attention to two approaches to the mathematical modelling of dry friction. Some authors formulate friction descriptions with the aid of postulates being a generalization of experimental facts. The well-known Coulomb friction model, a subdifferential friction equation proposed by Moreau (1979) and friction laws given by Rabier *et al.* (1986) belong to this group. The second approach gives friction equations deduced from an analysis of elastic and plastic deformations of surface asperity models [e.g. Halling (1976), Heilmann and Rigney (1981), Phan-Thien (1981) and Challen *et al.* (1987)]. Simple geometrical solids (rods, cubes, hemispheres, cones, pyramids, ellipsoids, etc.) were applied in the asperity modelling.

Dry friction which depends on the sliding direction is called anisotropic friction. A deviation in the friction force from the direction of sliding and a dependence of the friction magnitude on the sliding direction are features of contacts with frictional anisotropy.

The primary objective of this study is to prepare mathematical models of anisotropic friction and to investigate their properties. These models should include a full range of anisotropic friction behavior, i.e. centrosymmetric and non-centrosymmetric effects. It is solved by the generalization of a phenomenological model of anisotropic friction proposed by Zmitrowicz (1981, 1988, 1989).

2. EXPERIMENTAL RESULTS

Anisotropic friction is the actual phenomenon. There are many experimental studies which contribute to anisotropic friction and wear. Some of them are described by Zmitrowicz (1989). We include here those studies which were omitted in the previous work, and those which seem to us to be more important.

Friction depends on the sliding direction as a result of anisotropic surface roughness. It has been investigated experimentally by Rabinowicz (1957), Halaunbrenner (1960), Zieliński (1964), Eiss (1985), Felder (1988). Rabinowicz (1957) and Halaunbrenner (1960) realized experimental investigations of the friction force component normal to the sliding direction. This component occurs when the friction force direction is different from the motion direction. The surface topography is very important at the beginning of a test. As a result of wear, the surface roughness and frictional anisotropy become modified throughout the duration of sliding.

In substances consisting of more than one component, the surface has a mosaic structure since different components are exposed at different points on the surface. The anisotropy of solid surfaces due to their structure has been proved in many ways. The anisotropy of friction and wear appears essentially in alloys, crystalline, fibrous, reinforced and composite materials. It was observed by Tabor and Williams (1961), Hen-Won Chang (1983), Herold-Schmidt and Hinsberger (1987), Ciring *et al.* (1988), Vasiliyeva and Tofpenets (1989), Jacobs *et al.* (1990) and Kadijk and Broese van Groenou (1990). A large body of literature is devoted to experimental observations of anisotropic friction and wear of polymers and ceramics.

Crystal faces generally have different properties in different directions; hence the following phenomena are anisotropic: friction (Dyer, 1962; Bowden *et al.*, 1964; Childs, 1969; Casey and Wilks, 1973; Enomoto and Tabor, 1980; Buckley and Miyoshi, 1984; Miyoshi and Buckley, 1985), wear (Buckley and Miyoshi, 1984; Miyoshi and Buckley, 1985) and hardness (Brookes and Green, 1979; Brookes, 1981; Szymański and Szymański, 1989). The frictional anisotropy has been measured for a single crystal of diamond, copper, rutile, SiC, magnesium oxide, lithium fluoride, cobalt, beryllium, rhenium, titanium, aluminium, iron, sapphire, etc.

Investigations show that friction and wear depend on the crystallographic plane and on the sliding direction with respect to the crystallographic system. From the experiments carried out by Casey and Wilks (1973) it is known that the friction of diamond is strongly dependent on the direction of sliding, varying by up to 300% for (100) faces. In the case of the (001) face the friction is highest in the [100] directions and lowest in [110] directions. The face (001) shows four-fold symmetry, although this can be reduced to two-fold by polishing (Casey and Wilks, 1973).

The frictional anisotropy was observed in a rolling contact. Halaunbrenner (1958) experimentally investigated a dependence of the rolling resistance of a steel cylinder on motion direction. Two types of base surfaces were applied: with anisotropic roughness and with anisotropic mechanical properties (wood and NaCl monocrystal). Furthermore, anisotropic friction was investigated when a sapphire ball was rolled in various directions on the cube face of a copper crystal, Dyer (1962). The variation in rolling resistance with direction was 32%.

3. ANISOTROPIC FRICTION MODELS

Anisotropic friction phenomena are studied and recognized on different levels of generalization. First trials of the anisotropic friction description are given by Vantorin (1964), Michałowski and Mróz (1978), Curnier (1984), Felder (1986), Goyal and Ruina (1988).

According to the thermodynamical formulation of constitutive equations for friction (Zmitrowicz, 1987), the friction force vector $\mathbf{t}$ can be a function of the sliding velocity unit vector $\mathbf{v}$ (called a sliding direction) and the normal pressure N, e.g.

$$\mathbf{t} = -Nf(\mathbf{v}), \tag{1}$$

where

$$\mathbf{t} = t^i\mathbf{k}_i \in \underset{1}{\varepsilon_2}, \quad i = 1, 2, \tag{2}$$

$$\mathbf{v} = v^j\mathbf{e}_j \in \underset{2}{\varepsilon_2}, \quad j = 1, 2, \tag{3}$$

$$|\mathbf{v}| = 1, \tag{4}$$

$$N \in \mathbb{R}^+. \tag{5}$$

$\underset{1}{\varepsilon_2}$ and $\underset{2}{\varepsilon_2}$ are two two-dimensional vector spaces and $\{\mathbf{k}_i\}$ is an orthogonal basis of unit vectors in $\underset{1}{\varepsilon_2}$ and $\{\mathbf{e}_j\}$ is an arbitrary unit vector basis in $\underset{2}{\varepsilon_2}$ (Fig. 1).

A linear model of anisotropic friction is defined by the following equation:

$$\mathbf{t} = -N\mathbf{C}_1\mathbf{v}. \tag{6}$$

A non-linear model can be given by a polynomial function of the slip velocity unit vector

$$\mathbf{t} = -N(\mathbf{C}_1\mathbf{v} + \mathbf{C}_2\mathbf{v}^3 + \cdots + \mathbf{C}_n\mathbf{v}^{2n-1}), \tag{7}$$

where

$$\mathbf{C}_n\mathbf{v}^{2n-1} \equiv \mathbf{C}_n \cdot \underbrace{(\mathbf{v} \otimes \mathbf{v} \otimes \ldots \otimes \mathbf{v})}_{2n-1 \text{ copies}} \equiv \underbrace{[(\mathbf{C}_n\mathbf{v})\mathbf{v}\ldots]}_{2n-1 \text{ copies}}\mathbf{v}. \tag{8}$$

Friction tensors $\mathbf{C}_1, \mathbf{C}_2, \ldots, \mathbf{C}_n$ define the frictional anisotropy of contact:

$$\mathbf{C}_1 = C^{ij}\mathbf{k}_i \otimes \mathbf{e}_j, \tag{9}$$

$$\mathbf{C}_2 = C^{ijkl}\mathbf{k}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l, \tag{10}$$

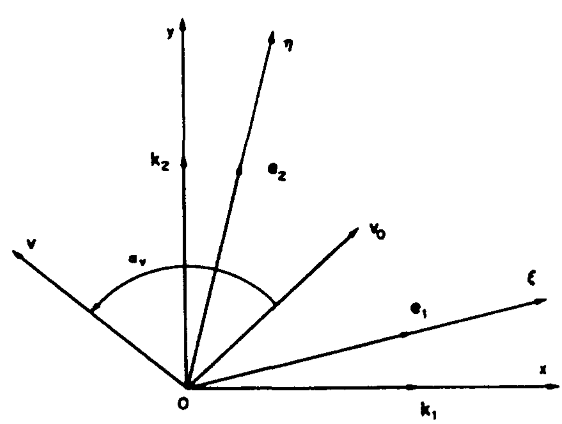

Fig. 1. Unit vectors of bases of the reference systems.

$$\mathbf{C}_n = C^{ij\ldots s}\mathbf{k}_i \otimes \underbrace{\mathbf{e}_j \otimes \ldots \otimes \mathbf{e}_s}_{2n-1 \text{ copies}}, \tag{11}$$

$$i, j, k, l, \ldots, s = 1, 2. \tag{12}$$

They are elements of linear spaces τ_{2i} formed by the tensor product of the space $\underset{1}{\varepsilon_2}$ and $(2i-1)$ times space $\underset{2}{\varepsilon_2}$, i.e.

$$\mathbf{C}_i \in \tau_{2i} = \underset{1}{\varepsilon_2} \otimes \underbrace{\underset{2}{\varepsilon_2} \otimes \ldots \otimes \underset{2}{\varepsilon_2}}_{2i-1 \text{ copies}}, \tag{13}$$

$$i = 1, 2, \ldots, n. \tag{14}$$

According to the objectivity axiom only elements with odd numbers of $\mathbf{v}$ may be included in the polynomial (7).

All linear and non-linear models of anisotropic friction described by eqns (6) and (7) are centrosymmetric.

It is easy to imagine a contact with non-equivalent motions to both sides in the same direction. A distribution of elements constructing a surface can make motion easy on one side of the given direction and difficult on the opposite side. In other words, there are differences in forwards and backwards sliding. The phenomenon is familiar to our daily experience. A physical non-equivalence on both sides of the sliding direction gives friction non-symmetry with respect to the sense of the direction (non-centrosymmetric case). For linear and non-linear friction models (6) and (7) both senses of the sliding direction are equivalent.

In the non-centrosymmetric case a change of sense of the sliding direction must be connected with a variation in anisotropy description. An introduction of friction tensors with components depending on the sliding direction makes it possible to define non-centrosymmetric anisotropy.

Let us assume that the friction tensors are "sliding direction-dependent". Thus, we postulate that the friction tensors are functions of the sliding direction parameter α_v, i.e.

$$\mathbf{C}_i = \mathbf{C}_i(\alpha_v), \quad i = 1, 2, \ldots, n, \tag{15}$$

where

$$\alpha_v \in \langle 0, 2\pi \rangle. \tag{16}$$

α_v is a measure of an oriented angle between the unit vector $\mathbf{v}_0$ of a reference direction in $\underset{2}{\varepsilon_2}$ and the slip velocity unit vector $\mathbf{v}$ in $\underset{2}{\varepsilon_2}$. Then, components of the friction tensors depend on the sliding direction with respect to the chosen reference direction.

Let us consider the friction tensor (15) which is a trigonometrical polynomial of the variable α_v with constant tensors $\mathbf{C}_{ik}$,

$$\mathbf{C}_i(\alpha_v) = \mathbf{C}_{i0} + \mathbf{C}_{i1}\cos(n_i\alpha_v) + \mathbf{C}_{i2}\sin(m_i\alpha_v), \tag{17}$$

$$n_i, m_i = 0, 1, 2, 3, \ldots, \tag{18}$$

$$\mathbf{C}_{ik} \in \tau_{2i} = \underset{1}{\varepsilon_2} \otimes \underbrace{\underset{2}{\varepsilon_2} \otimes \ldots \otimes \underset{2}{\varepsilon_2}}_{2i-1 \text{ copies}}, \tag{19}$$

$$k = 0, 1, 2, \quad i = 1, 2, \ldots, n. \tag{20}$$

The function (17) is single-valued and has finite values for all arguments α_v from the set (16).

Substituting eqn (17) into (7), we obtain the following constitutive equation of anisotropic friction:

$$\mathbf{t} = -N\{[\mathbf{C}_{10}+\mathbf{C}_{11}\cos(n_1\alpha_v)+\mathbf{C}_{12}\sin(m_1\alpha_v)]\mathbf{v}+\cdots+[\mathbf{C}_{n0}+\mathbf{C}_{n1}\cos(n_n\alpha_v) + \mathbf{C}_{n2}\sin(m_n\alpha_v)]\cdot\underbrace{(\mathbf{v}\otimes\ldots\otimes\mathbf{v})}_{2n-1\text{ copies}}\}. \quad (21)$$

The parameters n_i, m_i in eqn (21) may be taken as arbitrary. Symmetry properties and some physical assumptions will restrict elements of the polynomial (21).

Properties of constant tensors can be investigated easily. It is difficult if the tensors have coefficients depending on the sliding direction.

4. PROPERTIES OF ANISOTROPIC FRICTION

The total anisotropic friction coefficient and the angle of friction force inclination for any sliding direction can be obtained from the following relations:

$$\mu_\alpha = N^{-1}|\mathbf{t}|, \quad (22)$$

$$\sin\beta = \frac{\mathbf{t}\cdot\mathbf{v}^\perp}{|\mathbf{t}|}, \quad (23)$$

where, $\mathbf{v}^\perp$ is a unit vector orthogonal to the sliding direction

$$\mathbf{v}\cdot\mathbf{v}^\perp = 0, \quad |\mathbf{v}^\perp| = 1. \quad (24)$$

Friction coefficients of the friction force components collinear with the sliding direction and normal to it are given by

$$\mu_\alpha^\parallel = -N^{-1}\mathbf{t}\cdot\mathbf{v}, \quad (25)$$

$$\mu_\alpha^\perp = N^{-1}\mathbf{t}\cdot\mathbf{v}^\perp. \quad (26)$$

A curve drawn by the friction force vectors attached to the origin of the coordinate system is called the hodograph of the friction force. The shape of the curve can be studied by finding friction forces in all sliding directions. Some authors assume (by analogy to the theory of plasticity) that closed and convex curves with the normality condition for sliding direction contain all the information about frictional anisotropy. In this paper we assume that the hodograph is represented by closed and non-intersecting curves. In a particular case (unidirectional friction) the hodograph reduces to a segment of line.

The constitutive equation of friction (21) has the following properties.

Property 1

The friction force equation (21) satisfies the axiom of material objectivity.

A verification of material objectivity must relate to the adequate transformation of $\mathbf{v}$ and the sliding direction parameter α_v. Equation (21) has an invariant form with respect to arbitrary transformation from the full orthogonal group $\mathcal{O}$, if it obeys the following condition:

$$\mathbf{t}(\mathbf{R}\mathbf{v}, N) = \mathbf{R}\mathbf{t}(\mathbf{v}, N), \quad \forall\, \mathbf{R}\in\mathcal{O}, \quad (27)$$

where

$$\mathbf{R}^{-1} = \mathbf{R}^T, \quad \det \mathbf{R} = \pm 1, \tag{28}$$

$$\mathbf{t}(\mathbf{R}\mathbf{v}, N) = -N\{[\mathbf{C}_{10}+\mathbf{C}_{11}\cos(n_1\tilde{\alpha}_v)+\mathbf{C}_{12}\sin(m_1\tilde{\alpha}_v)]\mathbf{R}\mathbf{v}+\ldots\}. \tag{29}$$

$\tilde{\alpha}_v$ is a measure of an oriented angle between vectors $\mathbf{R}\mathbf{v}_0$ and $\mathbf{R}\mathbf{v}$. The orthogonal transformation preserves angles. Then, the angle between vectors $\mathbf{R}\mathbf{v}_0$ and $\mathbf{R}\mathbf{v}$ is equal to the angle between $\mathbf{v}_0$ and $\mathbf{v}$. Thus, eqn (21) obeys the condition of material objectivity (27). Notice that the constant friction tensors must be transformed in the general anisotropic friction case.

Property 2

Any friction tensor $\mathbf{C}_i(\alpha_v)$ is positive definite.

From the Second Law of Thermodynamics it follows that in every case of frictional contact the power of the friction force is non-positive

$$\mathbf{t}\cdot\boldsymbol{v} \leqslant 0, \quad \forall \boldsymbol{v}, \tag{30}$$

where, $\boldsymbol{v}$ is the sliding velocity. Substituting eqn (21) into (30) and taking into account that N and $|\boldsymbol{v}|$ are positive and that

$$\mathbf{v} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|} \tag{31}$$

we obtain the following condition

$$\boldsymbol{v}^T[\mathbf{C}_{10}+\mathbf{C}_{11}\cos(n_1\alpha_v)+\mathbf{C}_{12}\sin(m_1\alpha_v)]\boldsymbol{v}+\cdots$$
$$+\underbrace{\boldsymbol{v}^T\{\ldots\boldsymbol{v}^T}_{n\text{ copies}}[\mathbf{C}_{n0}+\mathbf{C}_{n1}\cos(n_n\alpha_v)+\mathbf{C}_{n2}\sin(m_n\alpha_v)]\underbrace{\boldsymbol{v}\ldots\}\boldsymbol{v}}_{n\text{ copies}} \geqslant 0. \tag{32}$$

We can replace the inequality (32), e.g. by the following restrictions on the friction tensors $\mathbf{C}_i(\alpha_v)$:

$$\boldsymbol{v}^T[\mathbf{C}_{10}+\mathbf{C}_{11}\cos(n_1\alpha_v)+\mathbf{C}_{12}\sin(m_1\alpha_v)]\boldsymbol{v} \geqslant 0, \tag{33}$$

$$\underbrace{\boldsymbol{v}^T\{\ldots\boldsymbol{v}^T}_{n\text{ copies}}[\mathbf{C}_{n0}+\mathbf{C}_{n1}\cos(n_n\alpha_v)+\mathbf{C}_{n2}\sin(m_n\alpha_v)]\underbrace{\boldsymbol{v}\ldots\}\boldsymbol{v}}_{n\text{ copies}} \geqslant 0, \tag{34}$$

for every $\boldsymbol{v}$ and $\alpha_v \in \langle 0, 2\pi\rangle$. Trigonometrical functions $\cos(n_i\alpha_v)$ and $\sin(m_i\alpha_v)$ have positive and negative values for $\alpha_v \in \langle 0, 2\pi\rangle$. Thus, conditions (33) and (34) are satisfied for all α_v depending on relations between values of components of the constant tensors $\mathbf{C}_{1k}, \ldots, \mathbf{C}_{nk}$ ($k = 0, 1, 2$).

Property 3

The constitutive equation (21) determines anisotropic friction with an arbitrary number of principal directions. Principal directions with different friction values depending on the sense of the sliding direction and unidirectional principal directions can be described by eqn (21).

These directions are called principal directions of friction which satisfy the following condition:

$$\mathbf{t}\cdot\mathbf{v}^{\perp} = 0. \tag{35}$$

The friction force component normal to the principal direction and the inclination angle are equal to zero

$$\mu_2^{\perp} = 0, \tag{36}$$

$$\beta = 0. \tag{37}$$

Let the base vectors $\{\mathbf{k}_i\}$ and $\{\mathbf{e}_j\}$ coincide with the orthogonal reference system Oxy, and the reference direction $\mathbf{v}_0$ coincide with the Ox axis. Then, the velocity unit vector components given with respect to this basis are

$$[\mathbf{v}] = [\cos\alpha_r \quad \sin\alpha_r]^{\mathrm{T}}. \tag{38}$$

The unit vector orthogonal to the slip direction is

$$[\mathbf{v}^{\perp}] = [\sin\alpha_r \quad -\cos\alpha_r]^{\mathrm{T}}. \tag{39}$$

Equation (21) has the following form in representation notation, if we restrict ourselves to the second order tensors only:

$$t^i = -N[C_0^{ij} + C_1^{ij}\cos(n_1\alpha_r) + C_2^{ij}\sin(m_1\alpha_r)]v_j. \tag{40}$$

After substitution of the components (39) into the equation of principal directions (35), we obtain

$$t^1\sin\alpha_r - t^2\cos\alpha_r = 0. \tag{41}$$

Substituting the unit vector components (38) and the friction force components (40) into this relation yields

$$\begin{aligned} &C_0^{12}\sin^2\alpha_r - C_0^{21}\cos^2\alpha_r + (C_0^{11} - C_0^{22})\sin\alpha_r\cos\alpha_r + [C_1^{12}\sin^2\alpha_r - C_1^{21}\cos^2\alpha_r \\ &\quad + (C_1^{11} - C_1^{22})\sin\alpha_r\cos\alpha_r]\cos(n_1\alpha_r) + [C_2^{12}\sin^2\alpha_r \\ &\quad - C_2^{21}\cos^2\alpha_r + (C_2^{11} - C_2^{22})\sin\alpha_r\cos\alpha_r]\sin(m_1\alpha_r) = 0. \end{aligned} \tag{42}$$

The number of principal directions defined by eqn (42) depends on the parameters n_1, m_1. If $n_1 = 1$ or $m_1 = 1$ then unidirectional principal directions may exist and the trigonometrical equation (42) describes at most four principal directions. If $n_1 = 4$ or $m_1 = 4$ then eqn (42) determines at most six principal directions (there are no unidirectional principal directions). Using isotropic second order constant tensors

$$C_0^{ij} = \mu_0\delta^{ij}, \quad C_1^{ij} = \mu_1\delta^{ij}, \quad C_2^{ij} = \mu_2\delta^{ij}, \tag{43}$$

where, μ_0, μ_1, μ_2 are friction coefficients, δ^{ij} is the Kronecker delta, and substituting them into (42) we get the equation satisfied for any $\alpha_r \in \langle 0, 2\pi\rangle$. It means that all sliding directions are principal directions independent of the values of the parameters n_1, m_1. Generally, arbitrary numbers of principal directions can be described by eqn (42).

It has been shown (Zmitrowicz, 1989) that the constitutive equation with constant friction tensors (7) can define the number of principal directions equal to the highest order of the friction tensors. In the case of the constitutive equation (21) the number of principal directions depends on the order of the constant tensors $\mathbf{C}_{ik}$ and values of parameters n_i, m_i. If n_i, m_i are even numbers then the number of principal directions is at most equal to a sum of the highest tensor order and a maximum from $\{n_i, m_i\}$. If n_i, m_i are odd numbers then unidirectional principal directions may exist and the number of principal directions is at most equal to a sum of the highest tensor order and a maximum from $\{2n_i, 2m_i\}$. The unidirectional principal directions satisfy the condition (35) on one side only, i.e. for $\mathbf{v}$ and not for $-\mathbf{v}$.

Notice that a greater number of principal directions can be described by the constitutive equation with tensors depending on the sliding direction than by the equation with constant tensors of a given order.

Property 4

If the parameters n_i, m_i are odd numbers then the constitutive equation (21) describes non-centrosymmetric anisotropic friction.

The inversion $-\mathbf{1}$ is that transformation which characterizes anisotropy having central symmetry. There are frictional anisotropies whose symmetry does not imply that $-\mathbf{1}$ be in a group of symmetry.

In the case of the constitutive equation (21), we investigate the inversion transformation with respect to constant tensors $\mathbf{C}_{ik}$ ($i = 1, \ldots, n$; $k = 0, 1, 2$) and trigonometrical functions of the sliding direction parameter α_v.

All even order constant tensors $\mathbf{C}_{ik}$ are invariant with respect to the inversion, i.e.

$$\left[\overset{2i}{\underset{1}{\otimes}}(-\mathbf{1})\right]\cdot\mathbf{C}_{ik} = \mathbf{C}_{ik}, \tag{44}$$

where

$$\mathbf{C}_{ik} = C_k^{jl\ldots s}\mathbf{k}_j \otimes \mathbf{e}_l \otimes \ldots \otimes \mathbf{e}_s, \tag{45}$$

$$\left[\overset{2i}{\underset{1}{\otimes}}(-\mathbf{1})\right]\cdot\mathbf{C}_{ik} \equiv C_k^{jl\ldots s}(-\mathbf{1})\mathbf{k}_j \otimes (-\mathbf{1})\mathbf{e}_l \otimes \ldots \otimes (-\mathbf{1})\mathbf{e}_s, \tag{46}$$

$$i = 1, 2, \ldots, n, \quad k = 0, 1, 2, \quad j, l, \ldots, s = 1, 2. \tag{47}$$

In the three-dimensional space, the inversion is equivalent to a rotation of π about a given axis and next to the reflection with respect to the plane orthogonal to this axis. For the two-dimensional space the inversion reduces to the rotation about the normal $\mathbf{n}$ to the space of the angle π

$$-\mathbf{1} \equiv \mathbf{R}_{\mathbf{n}}^{\pi}. \tag{48}$$

Tensor $\mathbf{R}_{\mathbf{n}}^{\pi}$ denotes a rotation of π radians about the unit vector $\mathbf{n}$. Thus the following conditions determine the inversion transformation for the trigonometrical functions:

$$\forall \alpha_v \in \langle 0, 2\pi \rangle : \begin{cases} \cos\left[n_i(\alpha_v + \pi)\right] = \cos(n_i\alpha_v), \\ \sin\left[m_i(\alpha_v + \pi)\right] = \sin(m_i\alpha_v). \end{cases} \tag{49}$$

The trigonometrical functions are symmetric with respect to the center of symmetry if n_i, m_i are even numbers. The central symmetry does not exist if n_i, m_i are odd numbers.

Property 5

Anisotropic friction description (21) defines an infinite number of neutral directions if n_i, m_i are even numbers. It has a finite number of neutral directions if n_i, m_i are odd numbers.

The particular sliding direction $\mathbf{v}$ which satisfies the condition

$$\mathbf{t}(\mathbf{v}, N) = -\mathbf{t}(-\mathbf{v}, N) \tag{50}$$

we call the neutral direction of friction. It has the same friction properties (i.e. the friction coefficient and the angle of the friction force deviation) for both sides of the direction $\mathbf{v}$. If the sliding direction is defined by the angle α_v, then

$$\mu_2(\alpha_v) = \mu_2(\alpha_v + \pi), \tag{51}$$

$$\beta(\alpha_v) = \beta(\alpha_v + \pi). \tag{52}$$

Centrosymmetric anisotropic friction has an infinite number of neutral directions but non-centrosymmetric friction has a finite number of these directions. Thus, the constitutive equation (21) describes: a finite number of neutral directions if n_i, m_i are odd numbers, an infinite number of neutral directions if n_i, m_i are even numbers. Linear and non-linear models with constant friction tensors (6), (7) have an infinite number of neutral directions in any case.

Property 6

The constitutive equation (21) restricted to single trigonometrical function sine or cosine and n_i, m_i equal to odd numbers determines frictional anisotropy with a finite, odd number of neutral directions. Using two or more trigonometrical functions we can define anisotropy with an even number of neutral directions.

The constant tensors $\mathbf{C}_{ik}$ of the constitutive equation (21) are of even order. They define an infinite number of neutral directions independent of the type of tensor. It implies that the number of neutral directions of the constitutive equation is determined by the trigonometrical functions only. A finite number of neutral directions exists for n_i, m_i equal to odd numbers. The trigonometrical functions with odd parameters n_i, m_i have values with different sense for arguments α_v and $(\alpha_v + \pi)$, where $\alpha_v \in \langle 0, 2\pi \rangle$. Thus, zeros of the trigonometrical functions are values of the functions equal for both arguments α_v and $(\alpha_v + \pi)$. We conclude that the number of neutral directions is equal to the odd parameters n_i, m_i, if the constitutive equation has a single trigonometrical function.

Neutral directions for the constitutive equation with two or more trigonometrical functions are different than those for the single trigonometrical function. Using two or more trigonometrical functions of different type in the constitutive equation we can get an even number of neutral directions.

Property 7

Identity, inversion, rotations and mirror reflections are elements of symmetry groups of the friction tensors $\mathbf{C}_i(\alpha_v)$.

Symmetry properties are described by specifying a set of linear transformations which map anisotropic friction in the reference state onto an equivalent state. Also a composition of two transformations must satisfy this general rule. The set of all transformations representing different symmetries forms a mathematical group. The symmetry groups give the precise mathematical description of the symmetry properties of anisotropic friction. The constitutive equation (21) defining anisotropic friction of a contact possessing symmetric properties is invariant under the group of transformations describing the symmetry of the friction.

Thanks to symmetry the number of unknown parameters of the constitutive description and a number of necessary exerimental measurements can be reduced.

The group of symmetry of the constitutive polynomial is the intersection of the symmetry groups of the tensors $\mathbf{C}_i(\alpha_v)$, $i = 1, 2, \ldots, n$

$$\mathcal{G}_t = \mathcal{G}(\mathbf{C}_1) \cap \mathcal{G}(\mathbf{C}_2) \cap \ldots \cap \mathcal{G}(\mathbf{C}_n). \tag{53}$$

In the case of the constitutive equation (21), we investigate groups of symmetry with respect to constant tensors $\mathbf{C}_{ik}$ and the trigonometrical functions of the sliding direction parameter α_v. The symmetry group of elements of the constitutive polynomial is the intersection of the symmetry groups of the constant tensors and the trigonometrical functions, i.e.

$$\mathcal{G}(\mathbf{C}_{i1} \cos n_i \alpha_v) = \mathcal{G}(\mathbf{C}_{i1}) \cap \mathcal{G}(\cos n_i \alpha_v), \tag{54}$$

$$\mathcal{G}(\mathbf{C}_{i2} \sin m_i \alpha_v) = \mathcal{G}(\mathbf{C}_{i2}) \cap \mathcal{G}(\sin m_i \alpha_v). \tag{55}$$

The subgroup $\mathscr{G}(\mathbf{C}_{ik})$ of the full orthogonal group $\mathcal{C}$ is called a symmetry group of the constant tensor $\mathbf{C}_{ik}$ if it satisfies the following relation:

$$\mathscr{G}(\mathbf{C}_{ik}) = \left\{\mathbf{R}:\mathbf{R}\in\mathcal{C},\left(\overset{2i}{\underset{1}{\otimes}}\mathbf{R}\right)\cdot\mathbf{C}_{ik} = \mathbf{C}_{ik}\right\}, \tag{56}$$

where

$$\left(\overset{2i}{\underset{1}{\otimes}}\mathbf{R}\right)\cdot\mathbf{C}_{ik} \equiv C_k^{jl\ldots s}\mathbf{Rk}_j\otimes\mathbf{Re}_l\otimes\ldots\otimes\mathbf{Re}_s. \tag{57}$$

The full orthogonal group $\mathcal{C}$ contains all orthogonal transformations satisfying (28).

We give definitions of symmetry of the trigonometrical functions with respect to the following transformations: inversion, rotations and mirror reflections. Let us first consider rotations. Functions $\cos(n_i\alpha_r)$ and $\sin(m_i\alpha_r)$ are invariant under the rotations of angle Φ and Ψ about normal $\mathbf{n}$ to the contact, if they satisfy the following conditions:

$$\exists\,\Phi\in\langle 0,2\pi\rangle:\forall\,\alpha_r\in\langle 0,2\pi\rangle,\cos[n_i(\alpha_r+\Phi)] = \cos(n_i\alpha_r), \tag{58}$$

$$\exists\,\Psi\in\langle 0,2\pi\rangle:\forall\,\alpha_r\in\langle 0,2\pi\rangle,\sin[m_i(\alpha_r+\Psi)] = \sin(m_i\alpha_r). \tag{59}$$

It implies that the rotation angles are related by

$$\Phi = \frac{2\pi}{n_i}, \tag{60}$$

$$\Psi = \frac{2\pi}{m_i}, \tag{61}$$

$$n_i, m_i \neq 0. \tag{62}$$

The trigonometrical functions satisfy the symmetry conditions for multiples of angles (60), (61), too. The rotations as elements of the symmetry group are denoted by

$$\mathbf{R}_{\mathbf{n}}^{\Phi}, \mathbf{R}_{\mathbf{n}}^{\Psi}. \tag{63}$$

Relations (60), (61) show that the rotation about $\mathbf{n}$ through the angle π or its multiple exists, if parameters n_i, m_i are even numbers. In these cases, the inversion $-\mathbf{1}$ is the element of the symmetry group and the description has a center of symmetry.

The mirror reflection is another element of the symmetry group. Let $\mathbf{J}_{\mathbf{m}_i}$ denote a mirror plane orthogonal to the axis $\mathbf{m}_i$. The definition of the mirror reflection is

$$\mathbf{J}_{\mathbf{m}_i} = -\mathbf{1}\mathbf{R}_{\mathbf{m}_i}^{\pi}. \tag{64}$$

According to (64) the mirror reflection is equivalent to a rotation ($\mathbf{R}_{\mathbf{m}_i}^{\pi}$) about the given direction $\mathbf{m}_i$ in the contact plane and to a rotation about the normal $\mathbf{n}$ to the plane ($\mathbf{R}_{\mathbf{n}}^{\pi} \equiv -\mathbf{1}$).

Let ε_i be an oriented angle between the reference direction unit vector $\mathbf{v}_0$ and the axis of the mirror reflection $\mathbf{m}_i$

$$\varepsilon_i \in \langle 0, 2\pi \rangle. \tag{65}$$

Then, the arbitrary sliding direction determined by the angle α_v after mirror reflection with respect to the axis $\mathbf{m}_i$ takes a position in the contact plane defined by the following angle

$$\tilde{\alpha}_v = \pi + 2\varepsilon_i - \alpha_v. \tag{66}$$

Thus, the trigonometrical functions cos $(n_i\alpha_v)$ and sin $(m_i\alpha_v)$ invariant with respect to mirror reflections regarding the mirror plane orthogonal to the direction ε_i must satisfy the following conditions:

$$\exists\, \varepsilon_i \in \langle 0, 2\pi \rangle : \forall\, \alpha_v \in \langle 0, 2\pi \rangle \begin{cases} \cos\,[n_i(\pi + 2\varepsilon_i - \alpha_v)] = \cos\,(n_i\alpha_v), \\ \sin\,[m_i(\pi + 2\varepsilon_i - \alpha_v)] = \sin\,(m_i\alpha_v). \end{cases} \tag{67}$$

Relations (67) are not equivalent to those for rotations, since there is no rotation which reproduces mirror reflection.

Using (67) we can determine axes of mirror reflections. The trigonometrical functions have symmetry with respect to mirror planes orthogonal to the following directions ε_i: for the function cos $(n_i\alpha_v)$

$$\begin{aligned} n_i &= 1, \quad \varepsilon_i = \pi/2, \\ n_i &= 2, \quad \varepsilon_i = \{0, \pi/2\}, \\ n_i &= 3, \quad \varepsilon_i = \{\pi/6, \pi/2, 5/6\pi\}\,; \end{aligned} \tag{68}$$

for the function sin $(m_i\alpha_v)$

$$\begin{aligned} m_i &= 1, \quad \varepsilon_i = 0, \\ m_i &= 2, \quad \varepsilon_i = \{\pi/4, 3/4\pi\}, \\ m_i &= 3, \quad \varepsilon_i = \{0, \pi/3, 2/3\pi\}. \end{aligned} \tag{69}$$

From the point of view of group theory, the identity **1** always belongs to the symmetry group of anisotropic friction.

All transformations from a given symmetry class of anisotropic friction must have one stationary point in the contact plane. In other words, all rotations and mirror planes must intersect at one point. It is obvious, since in the opposite case rotations about non-intersecting axes or reflections with respect to non-intersecting planes produce a translation of the contact area.

Property 8

In the case of non-centrosymmetric anisotropic friction, mirror reflections are with respect to those principal directions of friction which are collinear with neutral directions (denoted by $\mathbf{J}_{\mathbf{m}_i}$). If there is an infinite number of principal directions or there is no principal direction, then mirror reflections are with respect to a finite number of neutral directions (denoted by $\mathbf{J}_{\mathbf{u}_i}$).

There are examples of non-centrosymmetric anisotropic friction with principal directions but mirror reflections are with respect only to those directions which are simultaneously neutral. If there are no common principal and neutral directions then anisotropic friction has no mirror reflections. Centrosymmetric anisotropic friction has an infinite number of neutral directions, and the mirror reflections may be with respect to principal directions.

There are no mirror reflections with respect to principal directions having different friction values depending on the sense of the sliding direction, and with respect to unidirectional principal directions. These principal directions are not neutral directions.

Using isotropic tensors $\mathbf{C}_{ik}$ in the constitutive equation and n_i, m_i equal to odd numbers we obtain examples of anisotropic friction with a finite number of neutral directions. Mirror reflections are with respect to these directions.

Property 9

If in the case of centrosymmetric anisotropic friction there is an infinite number of principal directions or there is no principal direction, then mirror reflections are with respect to a finite number of extremal value directions (denoted by $\mathbf{J}_{s_i}$).

This sliding direction defined by the oriented angle $\alpha_r \in \langle 0, 2\pi \rangle$ we call the direction of anisotropic friction extremal value which gives for two functions $\mu_\alpha(\alpha_r)$ and $\mu_\alpha^\perp(\alpha_r)$ extremal values (minimum or maximum), simultaneously, i.e.

$$\mu_\alpha(\alpha_r) = \min\{\mu_\alpha(\tilde{\alpha}_r) ; \tilde{\alpha}_r \in \langle 0, 2\pi \rangle\},$$

or

$$\mu_\alpha(\alpha_r) = \max\{\mu_\alpha(\tilde{\alpha}_r) ; \tilde{\alpha}_r \in \langle 0, 2\pi \rangle\}, \tag{70}$$

and

$$\mu_\alpha^\perp(\alpha_r) = \min\{\mu_\alpha^\perp(\tilde{\alpha}_r) ; \tilde{\alpha}_r \in \langle 0, 2\pi \rangle\},$$

or

$$\mu_\alpha^\perp(\alpha_r) = \max\{\mu_\alpha^\perp(\tilde{\alpha}_r) ; \tilde{\alpha}_r \in \langle 0, 2\pi \rangle\}. \tag{71}$$

Using isotropic tensors $\mathbf{C}_{ik}$ in the constitutive equation and n_i, m_i equal to even numbers we obtain examples of anisotropic friction with an infinite number of principal and neutral directions. Then, mirror reflections are with respect to a finite number of extremal value directions.

In the case of isotropic friction all sliding directions are principal, neutral and extremal value directions. Mirror reflections are with respect to all sliding directions.

Property 10

If we restrict ourselves to the second and fourth order constant tensors $\mathbf{C}_{1k}$, $\mathbf{C}_{2k}$ ($k = 0, 1, 2$) then the following types of tensors can be distinguished with the aid of symmetry groups: isotropic, anisotropic, orthotropic, tetragonal anisotropic and axisymmetric anisotropic.

The restrictions imposed by the symmetry groups on the form of the second and fourth order constant tensors are described by Zmitrowicz (1981, 1989). Here, types of constant friction tensors are listed by name with transformations defining their symmetric properties. We have the following types of constant tensors:

isotropic

$$\mathscr{G}(\mathbf{C}_{ik}) = \{\mathscr{O}\}, \tag{72}$$

anisotropic

$$\mathscr{G}(\mathbf{C}_{ik}) = \{\pm \mathbf{1}\}. \tag{73}$$

orthotropic

$$\mathscr{G}(\mathbf{C}_{ik}) = \{\pm \mathbf{1}, \mathbf{J}_{\mathbf{m}_1}, \mathbf{J}_{\mathbf{m}_2}\}, \tag{74}$$

tetragonal anisotropic

$$\mathscr{G}(\mathbf{C}_{ik}) = \{+\mathbf{1}, \mathbf{R}_{\mathbf{n}}^{\pi}, \mathbf{J}_{\mathbf{m}_1}, \mathbf{J}_{\mathbf{m}_2}, \mathbf{J}_{\mathbf{m}_3}, \mathbf{J}_{\mathbf{m}_4}\}, \tag{75}$$

axisymmetric anisotropic

$$\mathscr{G}(\mathbf{C}_{ik}) = \{\mathbf{R}_{\mathbf{n}}^{\varphi}\}, \quad \varphi \in \langle 0, 2\pi \rangle, \tag{76}$$

where, $\mathbf{m}_i$ $(i = 1, \ldots, 4)$ are principal directions.

Since the linear and non-linear constitutive equations (6), (7) are polynomials, frictional anisotropies have the same symmetries and names as the constant friction tensors.

We extend the considerations presented by Zmitrowicz (1981, 1989) into detailed properties of axisymmetric anisotropic and unidirectional anisotropic constant tensors of the second order ($\mathbf{C}_{10}$). The axisymmetric anisotropic tensor defined by the group of symmetry (76) has the following properties: the friction coefficient of the normal component of the friction force, the total friction coefficient and the angle of inclination of the friction force to the sliding direction are constant for all sliding directions. There are no principal directions. In this case components of the second order friction tensor $\mathbf{C}_{10}$ are restricted by the following relations:

$$C_0^{11} = C_0^{22}, \quad C_0^{12} = -C_0^{21}. \tag{77}$$

The friction force vectors given for all sliding directions draw a circle with radius equal to

$$|\mathbf{t}| = N\mu_{\alpha}, \tag{78}$$

where

$$\mu_{\alpha} = \sqrt{(C_0^{11})^2 + (C_0^{12})^2}. \tag{79}$$

To avoid repetition, we have assumed in the previous works that the linear transformation (6) is non-singular (in the sense that its determinant is not equal to zero). The singular, second order friction tensor describes the case with definite friction in one principal direction and frictionless in the second principal direction (ideal smoothness). Both principal directions are orthogonal. Some authors call this particular case a unidirectional anisotropic friction (Vantorin, 1964).

Let us reject the assumption that the friction tensor $\mathbf{C}_{10}$ must be non-singular. Then, we have

$$\det \mathbf{C}_{10} = 0. \tag{80}$$

In this case, friction coefficients in two orthogonal principal directions are equal to

$$\mu_1 = 0, \tag{81}$$

$$\mu_2 = C_0^{11} + C_0^{22}. \tag{82}$$

The symmetry group of the unidirectional anisotropic friction is the same as for orthotropic friction (74). In the case of a singular tensor, the linear transformation (6) maps a unit circle into a segment of line. The friction force acts only in one direction. The function of the friction force inclination angle has a discontinuity point.

Property 11

Centrosymmetric anisotropic friction with groups of symmetry typical for orthotropic, anisotropic and tetragonal anisotropic friction and with hodographs of shapes other than simple curves (ellipse, circle), and anisotropic friction with a four-fold rotation axis can be described with the aid of the constitutive equation (21) and n_i, m_i equal to even numbers.

The constitutive equation (21) makes it possible to define types of frictional anisotropy other than those given by the linear and non-linear equations (6) and (7).

Orthotropic friction with hodographs other than an ellipse can be defined by the constitutive equation (21). This case occurs if we take orthotropic tensors $\mathbf{C}_{10}$, $\mathbf{C}_{11}$ and n_1 equal to even numbers. Then symmetries of orthotropic tensors can coincide with all symmetries (for $n_1 = 2$) or with some symmetries (for $n_1 > 2$) of the trigonometrical function.

Anisotropic friction with 0, 1, 2 principal directions and hodographs different to an ellipse can be described. We get this case using $\mathbf{C}_{10}$, $\mathbf{C}_{11}$ anisotropic with an adequate number of principal directions and n_1 equal to even numbers. The group of symmetry is of the type (73).

Furthermore, the constitutive equation (21) can realize tetragonal anisotropy with a hodograph shape other than that in the non-linear case (7) with the fourth order tensor. It occurs if we take tetragonal anisotropic fourth order tensors $\mathbf{C}_{20}$, $\mathbf{C}_{21}$ and $n_2 = 4, 8, \ldots$. Using $n_2 = 2$ we get the case where the trigonometrical function has fewer symmetries than the constant tensors $\mathbf{C}_{20}$, $\mathbf{C}_{21}$. Symmetries of the tetragonal anisotropic friction occur for $n_2 \geqslant 4$.

Anisotropic friction with a four-fold rotation axis is realized with the aid of tetragonal anisotropic tensors $\mathbf{C}_{20}$, $\mathbf{C}_{22}$ and the function sin $(4\alpha_v)$. In this case there are four principal directions but there are no mirror reflections. The symmetry group has identity and four-fold rotation axis

$$\mathscr{G}_t = \{+\mathbf{1}, \mathbf{R}_n^{\pi/2}\}. \tag{83}$$

The multiplication table (Cayley square) has the following form

	$\mathbf{1}$	$\pi/2$	$-\mathbf{1}$	$3/2\pi$
$\mathbf{1}$	$\mathbf{1}$	$\pi/2$	$-\mathbf{1}$	$3/2\pi$
$\pi/2$	$\pi/2$	$\mathbf{1}$	$3/2\pi$	$\mathbf{1}$
$-\mathbf{1}$	$-\mathbf{1}$	$3/2\pi$	$\mathbf{1}$	$\pi/2$
$3/2\pi$	$3/2\pi$	$\mathbf{1}$	$\pi/2$	$-\mathbf{1}$

(84)

This is an Abelian group. The rotation angles $\pi/2$, $3/2\pi$ and inversion $-\mathbf{1}$ define the elements of the subgroup of rotation about the four-fold rotation axis.

Anisotropic friction with an infinite number of principal directions and hodographs different to a circle is achieved using $\mathbf{C}_{ik}$ as isotropic. Its group of symmetry is a subgroup of the full orthogonal group

$$\mathscr{G}_t \subset \mathcal{O}. \tag{85}$$

All sliding directions are neutral directions for n_i, m_i equal to even numbers. Thus, mirror reflections are with respect to extremal value directions. Using isotropic second order tensors $\mathbf{C}_{10}$, $\mathbf{C}_{11}$ and $n_1 = 4$ we get friction with the following group of symmetry:

$$\mathscr{G}_t = \{+\mathbf{1}, \mathbf{R}_n^{\pi/2}, \mathbf{J}_{s_1}, \mathbf{J}_{s_2}, \mathbf{J}_{s_3}, \mathbf{J}_{s_4}\}. \tag{86}$$

The mirror reflections are with respect to four extremal value directions $\mathbf{s}_i$ $(i = 1, \ldots, 4)$. This is a special type of tetragonal anisotropic friction with an infinite number of principal directions.

Experimental measurements presented by Rabinowicz (1957) are an illustration of the orthotropic friction. Experiments made by Halaunbrenner (1960) are typical of anisotropic friction. A similarity of the tetragonal anisotropic friction with experimental investigations of frictional anisotropy for diamond crystals (Bowden *et al.*, 1964; Casey and Wilks, 1973; Enomoto and Tabor, 1980) and for rough surfaces (Zieliński, 1964) can be noticed.

Property 12

Non-centrosymmetric anisotropic friction with symmetry groups of different types, trigonal anisotropic friction, friction with an infinite number of principal directions and hodographs other than a circle can be described with the aid of the constitutive equation (21) and n_i, m_i equal to odd numbers.

Depending on the values of the parameters n_i, m_i and types of constant tensors $\mathbf{C}_{ik}$ we get frictional anisotropic descriptions with symmetry groups of different types. For n_i, m_i equal to odd numbers frictional anisotropies are non-centrosymmetric. Their groups of symmetry do not have the inversion element $-\mathbf{1}$.

In the simplest case the symmetry group of anisotropic friction has only the identity element

$$\mathscr{G}_t = \{+\mathbf{1}\}. \tag{87}$$

This group of symmetry occurs for the constitutive equation (21) restricted to the second order tensors, if n_1, m_1 are equal to odd numbers and tensors $\mathbf{C}_{1k}$ are without special symmetries. If tensors $\mathbf{C}_{1k}$ have symmetries which do not coincide with symmetries of trigonometrical functions, then we get the same result.

Another type of anisotropic friction has the symmetry group produced by identity and mirror reflections with respect to, e.g. principal direction $\mathbf{m}_1$:

$$\mathscr{G}_t = \{+\mathbf{1}, \mathbf{J}_{\mathbf{m}_1}\}. \tag{88}$$

This is an Abelian group and its Cayley square has the following form:

	$\mathbf{1}$	$\mathbf{J}_{\mathbf{m}_1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{J}_{\mathbf{m}_1}$.
$\mathbf{J}_{\mathbf{m}_1}$	$\mathbf{J}_{\mathbf{m}_1}$	$\mathbf{1}$

(89)

We arrive at this result using $\mathbf{C}_{10}$, $\mathbf{C}_{11}$ orthotropic and n_1 equal to odd numbers. Although, the fourth order tensors $\mathbf{C}_{20}$, $\mathbf{C}_{21}$ being tetragonal anisotropic and n_2 equal to odd numbers give the same group of symmetry. The mirror reflection is with respect to that principal direction which coincides with neutral direction.

A special type of trigonal anisotropic friction has the symmetry group produced by identity, three-fold rotation axis and three mirror reflections with respect to neutral directions $\mathbf{u}_i$ ($i = 1, 2, 3$)

$$\mathscr{G}_t = \{+\mathbf{1}, \mathbf{R}_{\mathbf{n}}^{2/3\pi}, \mathbf{J}_{\mathbf{u}_1}, \mathbf{J}_{\mathbf{u}_2}, \mathbf{J}_{\mathbf{u}_3}\}. \tag{90}$$

The group of symmetry (90) is a non-Abelian group of sixth order

	1	$2/3\pi$	$4/3\pi$	$\mathbf{J}_{u_1}$	$\mathbf{J}_{u_2}$	$\mathbf{J}_{u_3}$
1	1	$2/3\pi$	$4/3\pi$	$\mathbf{J}_{u_1}$	$\mathbf{J}_{u_2}$	$\mathbf{J}_{u_3}$
$2/3\pi$	$2/3\pi$	$4/3\pi$	1	$\mathbf{J}_{u_3}$	$\mathbf{J}_{u_1}$	$\mathbf{J}_{u_2}$
$4/3\pi$	$4/3\pi$	1	$2/3\pi$	$\mathbf{J}_{u_2}$	$\mathbf{J}_{u_3}$	$\mathbf{J}_{u_1}$
$\mathbf{J}_{u_1}$	$\mathbf{J}_{u_1}$	$\mathbf{J}_{u_2}$	$\mathbf{J}_{u_3}$	1	$2/3\pi$	$4/3\pi$.
$\mathbf{J}_{u_2}$	$\mathbf{J}_{u_2}$	$\mathbf{J}_{u_3}$	$\mathbf{J}_{u_1}$	$4/3\pi$	1	$2/3\pi$
$\mathbf{J}_{u_3}$	$\mathbf{J}_{u_3}$	$\mathbf{J}_{u_1}$	$\mathbf{J}_{u_2}$	$2/3\pi$	$4/3\pi$	1

(91)

This type of friction we derive using $\mathbf{C}_{10}$, $\mathbf{C}_{11}$ isotropic and $n_1 = 3$. It also acts for the fourth order isotropic tensors $\mathbf{C}_{20}$, $\mathbf{C}_{21}$ and $n_2 = 3$.

Using n_i, $m_i > 3$ we can obtain descriptions with groups of symmetry having new rotations and mirror reflections. Generally, the number of mirror reflections of the constitutive description with isotropic tensors $\mathbf{C}_{ik}$ and arbitrary n_i, m_i is equal to the number of mirror reflections of the trigonometrical functions. In all these cases the groups of symmetry are subgroups of the full orthogonal group (85). All sliding directions are principal directions and hodographs of friction force are of shapes other than a circle.

The effect of non-centrosymmetric anisotropic friction was experimentally observed for rough surfaces by Halaunbrenner (1960). Furthermore, non-centrosymmetric wear was observed by Kadijk and Broese van Groenou (1990).

5. A COMPOSITION OF TWO DIFFERENT FRICTIONAL ANISOTROPIES

Our analysis of the centrosymmetric and non-centrosymmetric anisotropic friction has been related to friction forces during the sliding of two contacting surfaces with single isotropic and anisotropic frictional properties. Now, we study frictional forces during the sliding of surfaces with different anisotropic friction properties.

Assume that there are orthogonal reference systems on both contacting surfaces. The coefficients of the constant friction tensors $\mathbf{C}_{ik}$ and parameters n_i, m_i can be determined experimentally by sliding a third body with isotropic friction properties on the surfaces of each contacting body. Thus, we get representations of the tensors $\overset{(1)}{\mathbf{C}}_{ik}$, $\overset{(2)}{\mathbf{C}}_{ik}$ and values of the parameters $\overset{(1)}{n}_i$, $\overset{(2)}{n}_i$, $\overset{(1)}{m}_i$, $\overset{(2)}{m}_i$. They describe friction properties of surfaces (1) and (2), respectively.

Relative positions of the contacting surfaces are described by an angle φ (Fig. 2). Then the following relation holds between the unit vectors of the bases of the reference systems

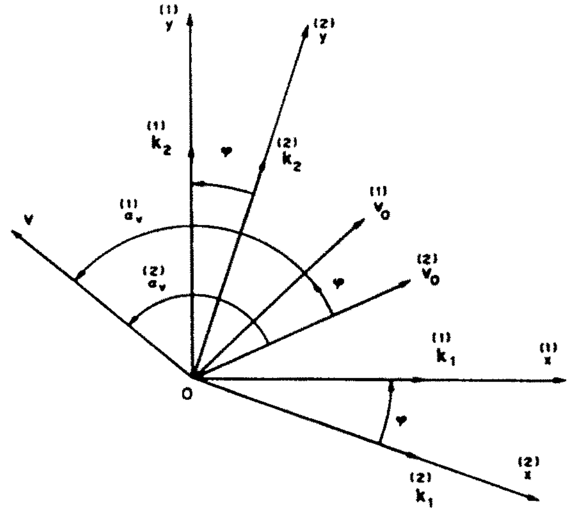

Fig. 2. Relative positions of the reference systems at the contact point of surfaces (1) and (2).

$$\overset{(2)}{\mathbf{k}}_i = B_i^j \overset{(1)}{\mathbf{k}}_j, \quad i,j = 1,2, \tag{92}$$

where

$$[B_i^j] \equiv \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \equiv [\mathbf{B}] \tag{93}$$

is an orthogonal rotation matrix.

We assume the reference directions $\overset{(1)}{\mathbf{v}}_0$ and $\overset{(2)}{\mathbf{v}}_0$ on both the contact surfaces. The angle φ determines relative positions of the reference directions (Fig. 2). Both unit vectors $\overset{(s)}{\mathbf{v}}_0$ ($s = 1, 2$) coincide at the initial position, i.e. for $\varphi = 0^\circ$. Then, the following relation holds between the sliding direction parameters $\overset{(1)}{\alpha}_v$ and $\overset{(2)}{\alpha}_v$ on the surfaces (1) and (2)

$$\overset{(2)}{\alpha}_v = \overset{(1)}{\alpha}_v + \varphi. \tag{94}$$

Let us assume that for a given normal pressure the friction force on the contact surface during relative sliding is equal to the product of a "composition coefficient" by the sum of the friction forces obtained for each surface taken separately, i.e.

$$\mathbf{t} = \kappa(\overset{(1)}{\mathbf{t}} + \overset{(2)}{\mathbf{t}}). \tag{95}$$

The forces $\overset{(1)}{\mathbf{t}}$ and $\overset{(2)}{\mathbf{t}}$ correspond to friction when sliding a third body with isotropic friction properties along the contacting surfaces. The composition coefficient κ is an experimental quantity and its value does not affect the description of the directional properties of friction.

The friction forces for surfaces (1) and (2) can be represented by

$$\overset{(s)}{\mathbf{t}} = -N[\overset{(s)}{\mathbf{C}}_1(\overset{(s)}{\alpha}_v)\mathbf{v} + \overset{(s)}{\mathbf{C}}_2(\overset{(s)}{\alpha}_v)\mathbf{v}^3 + \cdots + \overset{(s)}{\mathbf{C}}_n(\overset{(s)}{\alpha}_v)\mathbf{v}^{2n-1}], \quad s = 1,2, \tag{96}$$

where the friction tensors have the following form:

$$\overset{(s)}{\mathbf{C}}_i(\overset{(s)}{\alpha}_v) = \overset{(s)}{\mathbf{C}}_{i0} + \overset{(s)}{\mathbf{C}}_{i1}\cos(\overset{(s)}{n}_i \overset{(s)}{\alpha}_v) + \overset{(s)}{\mathbf{C}}_{i2}\sin(\overset{(s)}{m}_i \overset{(s)}{\alpha}_v), \quad i = 1,2,\ldots,n. \tag{97}$$

After substituting the friction forces (96) and the transformation relations (92) and (94) into eqn (95), we obtain the friction force relative to the contact of two different surfaces

$$\mathbf{t} = -N[\tilde{\mathbf{C}}_1(\alpha_v)\mathbf{v} + \tilde{\mathbf{C}}_2(\alpha_v)\mathbf{v}^3 + \ldots]. \tag{98}$$

According to the definition (95), the matrix representations of the friction tensors $\tilde{\mathbf{C}}_1(\alpha_v)$ and $\tilde{\mathbf{C}}_2(\alpha_v)$ are defined by

$$[\tilde{\mathbf{C}}_1(\alpha_v)] = \kappa\{\overset{(1)}{\mathbf{C}}_{10} + \overset{(1)}{\mathbf{C}}_{11}\cos(\overset{(1)}{n}_1\alpha_v) + \overset{(1)}{\mathbf{C}}_{12}\sin(\overset{(1)}{m}_1\alpha_v) + \mathbf{B}^{\mathrm{T}}\overset{(2)}{\mathbf{C}}_{10}\mathbf{B}$$
$$+ \mathbf{B}^{\mathrm{T}}\overset{(2)}{\mathbf{C}}_{11}\mathbf{B}\cos[\overset{(2)}{n}_1(\alpha_v+\varphi)] + \mathbf{B}^{\mathrm{T}}\overset{(2)}{\mathbf{C}}_{12}\mathbf{B}\sin[\overset{(2)}{m}_1(\alpha_v+\varphi)]\} \tag{99}$$

$$[\tilde{\mathbf{C}}_2(\alpha_v)] = \kappa\{\overset{(1)}{\mathbf{C}}_{20} + \overset{(1)}{\mathbf{C}}_{21}\cos(\overset{(1)}{n}_2\alpha_v) + \overset{(1)}{\mathbf{C}}_{22}\sin(\overset{(1)}{m}_2\alpha_v) + \mathbf{B}^{\mathrm{T}}(\mathbf{B}^{\mathrm{T}}\overset{(2)}{\mathbf{C}}_{20}\mathbf{B})\mathbf{B}$$
$$+ \mathbf{B}^{\mathrm{T}}(\mathbf{B}^{\mathrm{T}}\overset{(2)}{\mathbf{C}}_{21}\mathbf{B})\mathbf{B}\cos[\overset{(2)}{n}_2(\alpha_v+\varphi)] + \mathbf{B}^{\mathrm{T}}(\mathbf{B}^{\mathrm{T}}\overset{(2)}{\mathbf{C}}_{22}\mathbf{B})\mathbf{B}\sin[\overset{(2)}{m}_2(\alpha_v+\varphi)]\}. \tag{100}$$

The constant tensors $\overset{(s)}{\mathbf{C}}_{ik}$ transform according to the transformation rule for tensors, and the arguments of the trigonometrical functions transform with respect to the rule (94).

Let us consider some particular cases of the friction force for various contacts. Taking both surfaces with centrosymmetric anisotropic frictions we get a centrosymmetric anisotropic friction as a result of composition of these surfaces. The centrosymmetric friction in this case depends on the relative position of the contacting surfaces.

When the surface (1) has a non-centrosymmetric anisotropic friction and the surface (2) a centrosymmetric anisotropic friction, then frictional properties of the contact are non-centrosymmetric. A contact of the surface (1) having centrosymmetric friction properties and the surface (2) with non-centrosymmetric properties has a non-centrosymmetric anisotropic friction.

Generally the composition of two surfaces with non-centrosymmetric anisotropic friction properties gives a non-centrosymmetric anisotropic friction force. Although, for the given type of non-centrosymmetric friction and the relative position of the contacting surfaces we can obtain a centrosymmetric friction, in this case, too.

6. CONCLUSIONS

(1) Constitutive equations describing centrosymmetric (n_i, m_i even numbers) and non-centrosymmetric (n_i, m_i odd numbers) anisotropic friction are presented in this paper. Mathematical properties of the constitutive equations define a range of possible applications of the anisotropic friction models.

(2) Neutral and extremal value directions of friction have been defined. There are anisotropic frictions with a finite number (n_i, m_i odd numbers) and with an infinite number (n_i, m_i even numbers) of neutral directions.

(3) Symmetry groups of centrosymmetric and non-centrosymmetric anisotropic friction contain rotations and mirror reflections. Mirror reflections may be with respect to principal, neutral (n_i, m_i odd numbers) and extremal value (n_i, m_i even numbers) directions.

(4) The constitutive equations describe an arbitrary number of principal directions, and principal directions with different friction values depending on the sense of the direction, and unidirectional principal directions.

(5) The constitutive equations define orthotropic, anisotropic and tetragonal anisotropic friction force hodographs of shapes other than in the case of linear and non-linear models with constant friction tensors.

Frictional anisotropy can play a major role in engineering applications of monocrystals. The orientation of the crystal face is important in the case of diamond, corundum, SiC, etc. used in technological operations (e.g. in drawing, cutting, abrasive disks, wire drawing and long-life bearings). Special abrasive tools can be produced by adequate orientation and distribution of abrasive grains with respect to a workpiece.

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